5.5 Theorems About Roots of Polynomials

Rational Root Theorem

Any polynomial function with integer coefficients of the form
$P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0, \ldots$
has a limited number of possible roots of $P(x) = 0$

Integer roots must be factors of constant terms. Rational roots must have reduced form p/q where p is an integer factor of the constant and q is an integer factor of the leading coefficient.

All possible rational roots occur in the form of an organized list:

$$\frac{p}{q} = \frac{\pm \text{factors of constant}}{\pm \text{factors of lead coeff.}}$$

Ex 1. What are the rational roots of $2x^3 - x^2 - 27x + 36 = 0$

Factors of constant: $36 \rightarrow \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36$

Factors of coefficient: $2 \rightarrow \pm 1, \pm 2$

Possible factors: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36, \pm \frac{1}{2}, \pm 3/2, \pm 9/2$

Using the Rational Roots Theorem

1. Make list of possible rational roots
2. Find one root by using synthetic division
3. Continue finding roots and dividing until you have a second degree polynomial
4. Use any method for solving quadratics to find the remaining roots

Ex 2. What are the rational roots of $2x^3 + x^2 - 7x - 6 = 0$

$p \rightarrow 1, 2, 3, 6$

$q \rightarrow 1, 2$

$\frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$

$\boxed{x = -1, 2, -3/2}$

$$\begin{array}{r|rrrr} -1 & 2 & 1 & -7 & -6 \\ & \downarrow & -2 & 1 & 6 \\ \hline & 2 & -1 & -6 & 0 \end{array}$$

$2x^2 - x - 6$ -12 $-4 \; 3$

$2x^2 - 4x + 3x - 6$

$2x(x-2) + 3(x-2)$

$(2x+3)(x-2)$

Ex 3. What are the rational roots of $4x^3 + x^2 - 7x - 16 = 0$

$p \rightarrow 1, 2, 4, 8, 16$

$q \rightarrow 1, 2, 4$

$\frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \pm \frac{1}{2}, \pm \frac{1}{4}$

Conjugate Theorem	If $P(x)$ is a polynomial with rational coefficients, then the irrational roots of $P(x) = 0$ occur in conjugate pairs. That is, when $(a + \sqrt{b})$ is a root, then $(a - \sqrt{b})$ is also a root. Also, if $a + bi$ is a zero of a function, then $a - bi$ is also a zero.

Using the Conjugate Theorem	**Ex 4.** A cubic polynomial has real coefficients. If two of the roots are $x = 4$ and $x = \sqrt{5}$, what is the other root? $x = -\sqrt{5}$	**Ex 5.** If a quantic polynomial has real coefficients, and three of the zeros are $x = 2i, x = -3 + \sqrt{5}$ and $x = 1$, name the other zeros. $x = -2i \ , -3-\sqrt{5}$

Ex 6. Find a cubic polynomial equation with roots at $-2i$ and -5.

$x = -2i \qquad x = 2i \qquad x = -5$

$(x+2i)(x-2i)(x+5)$

$(x^2 \cancel{-2ix} \cancel{+2ix} -4i^2)(x+5)$

$+4$

$(x^2+4)(x+5)$

$\boxed{x^3+5x^2+4x+20}$

Ex 7. Find a polynomial function with a root at $4 + \sqrt{5}$

$x = 4+\sqrt{5} \qquad x = 4-\sqrt{5}$

$(x-4-\sqrt{5})(x-4+\sqrt{5})$

$x^2-4x+\cancel{\sqrt{5}x}-4x+16\cancel{-4\sqrt{5}}\cancel{-\sqrt{5}x}\cancel{+4\sqrt{5}}-5$

$\boxed{x^2-8x+11}$

Ex 8. Find a polynomial function with roots at $-5 - 7i$ and $2 - \sqrt{11}$

$x = -5-7i \quad x = -5+7i \quad x = 2-\sqrt{11} \quad x = 2+\sqrt{11}$

$[(x+5+7i)(x+5-7i)][(x-2+\sqrt{11})(x-2-\sqrt{11})]$

$(x^2+5x\cancel{-7ix}+5x+25\cancel{-35i}\cancel{+7ix}\cancel{+35i}-49i^2)(x^2-2x\cancel{-\sqrt{11}x}-2x+4\cancel{+2\sqrt{11}}\cancel{+\sqrt{11}x}\cancel{-2\sqrt{11}}-11)$

$+49$

$(x^2+10x+74)(x^2-4x-7)$

$x^4-4x^3-7x^2+10x^3-40x^2-70x+74x^2-296x-518$

$x^4+6x^3+27x^2-366x-518$