5.6 The fundamental Theorem of Algebra

The Fundamental Theorem of Algebra	If $P(x)$ is a polynomial of degree $n \geq 1$, then $P(x) = 0$ has exactly n roots, including multiple and complex roots. **Example:** $P(x) = 3x^3 - 2x + 3$ is a polynomial of degree 3, so $P(x) = 0$ has 3 roots.
Using the Fundamental Theorem	1. Write the polynomial in standard form. 2. Find the list of possible rational roots. 3. Use synthetic division to find roots. 4. Continue to factor until you have linear factors. 5. Use Fundamental Theorem to find the remaining roots.
	Ex. 1. What are the roots of the equation $x^4 + 2x^3 = 13x^2 - 10x$? $\pm 1, \pm 2, \pm 5, \pm 10$ $x^4 + 2x^3 - 13x^2 + 10x$ $x(x^3 + 2x^2 - 13x + 10)$ $1 \rfloor\ 1\quad 2\quad -13\quad 10$ $\downarrow\quad 1\quad 3\quad -10$ $1\quad 3\quad -10\quad 0$ $x^2 + 3x - 10$ $x^2 + 3x - 10$ $(x+5)(x-2)$ $x = -5, 2$ $x = 0$ $x = 1$ $x = -5$ $x = 2$ $x = 0, 1, -5, 2$
	Ex. 2. What are all of the zeros of the function $g(x) = 2x^4 - 3x^3 - x - 6$? $\pm 1, \pm 2, \pm 3, \pm 6, \pm 1/2, \pm 3/2$ $-1 \rfloor\ 2\quad -3\quad 0\quad -1\quad -6$ $\downarrow\quad -2\quad 5\quad -5\quad 6$ $2\quad -5\quad 5\quad -6\quad 0$ $2x^3 - 5x^2 + 5x - 6$ $2 \rfloor\ 2\quad -5\quad 5\quad -6$ $\downarrow\quad 4\quad -2\quad 6$ $2\quad -1\quad 3\quad 0$ not factorable $2x^2 - x + 3$ $\frac{1 \pm \sqrt{(-1)^2 - 4(2)(3)}}{2(2)}$ $\frac{1 \pm \sqrt{1-24}}{4}$ $\frac{1 \pm \sqrt{-23}}{4} = \frac{1}{4} \pm \frac{\sqrt{23}}{4} i$ $x = -1, 2, \frac{1}{4} \pm \frac{\sqrt{23}}{4} i$
Linear Factors	**Ex 3.** Find all the zeros of the function by factoring completely, into a product of linear factors. $f(x) = x^4 - 2x^3 - x^2 + 2x$ $x(x^3 - 2x^2 - x + 2)$ $x(x^2(x-2) - 1(x-2))$ $x(x^2-1)(x-2)$ $x(x+1)(x-1)(x-2)$ $x = 0, -1, 1, 2$

Practice

Find all the roots of the equation. $x^4 + 4x^3 + 7x^2 + 16x + 12 = 0$

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

$$\begin{array}{r|rrrrr} -1 & 1 & 4 & 7 & 16 & 12 \\ + & \downarrow & -1 & -3 & -4 & -12 \\ \hline & 1 & 3 & 4 & 12 & 0 \end{array}$$

Synthetic guess & check or grouping $\rightarrow$

$x^3+3x^2+4x+12$

$x^2(x+3)4(x+3)$

$(x^2+4)(x+3)$

$x^2+4=0$

$x^2=-4$

$x=\pm\sqrt{-4}$

$x=\pm 2i$

$x+3=0$

$x=-3$

$x=-1,-3,\pm 2i$

Find all of the solutions to $f(x) = x^4+2x^3+x^2-2x-2$

$p/q = \pm 1, \pm 2$

$$\begin{array}{r|rrrrr} 1 & 1 & 2 & 1 & -2 & -2 \\ + & \downarrow & 1 & 3 & 4 & 2 \\ \hline & 1 & 3 & 4 & 2 & 0 \end{array}$$

x^3+3x^2+4x+2

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 4 & 2 \\ + & \downarrow & -1 & -2 & -2 \\ \hline & 1 & 2 & 2 & 0 \end{array}$$

x^2+2x+2

$\frac{-2\pm\sqrt{4-4(1)(2)}}{2}$

$\frac{-2\pm\sqrt{-4}}{2}$

$\frac{-2\pm 2i}{2} = -1\pm i$

$\boxed{x=\pm 1, -1\pm i}$

Find all of the roots. $x^3 = 4x^2 - 8$

x^3-4x^2+8

$$\begin{array}{r|rrrr} 2 & 1 & -4 & 0 & 8 \\ + & \downarrow & 2 & -4 & -8 \\ \hline & 1 & -2 & -4 & 0 \end{array}$$

complete square or quad-formula $\rightarrow$ x^2-2x-4

$x^2-2x=4$

$x^2-2x+1=5$

$(x-1)^2=5$

$x-1=\pm\sqrt{5}$

$x=1\pm\sqrt{5}$

$x=2, 1\pm\sqrt{5}$

Can a fifth-degree polynomial with rational coefficients have 4 real roots and 1 irrational root? Explain why or why not?

no, irrational roots always come in conjugate pairs

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